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The generating set for some maximal soluble subgroups of GL(4,P^k)

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ABSTRACT: In this paper we show that the number of conjugacy classes of irreducible soluble GL(4,3) subgroup of is 108, whose :65 consist of imprimitive groups and 43 of primitive groups.

***Keywords*:** generating set, soluable, maximal

INTRODUCTION

In determining all permutation groups of a deferent degree The first work is probably due to Ruffini(1799)' who gives The possible orders of the groups of degree **5**.This them is taken up-again By Cauchy (1845)' who determines the possible orders of the groups of degree Up to **6**. Mathieu(1858)continus cauchy's work' determining the Possible Orders of the groups of degrees **7** and **8**. Miller(1896 a)Suggests that Ruffini's work is in Complate' and citesan omission in Cauchy's list for degree **6** .Serret(1850)determines all subgroups of S_4 and S_5.Miller(1896 a)says that thise work is correct.
kirkman(1862–3)Lists the transitive groups of degrees **3** to 10.Miller(1896 a) Says that This list is correct up to degree **7**' has missing six groups of Degree **8** ' another six of degree **9**' and is highly inaccurate for degree **10**.

Jordan(1871a)gives a table containing the numbers of conjugacy classes of primitive maximal soluble groups of degree lees then 10^6. He claims there are five such classes of degree **81**' but there are only four' (see to Chapter **6**).This error is Likely to lead to errors for larger degrees.Also' the Second And third entries in the last row of this table should be Swapped.In The same paper' Jordan also gives a table containing the numbers of conjugacy classes of transitive maximale soluble groups of degree Up to **10000**.(This is an a stounding achievment if for no other reason Than the amount of counting required to prepare it.for example' if p is a prime greater than **3**' Jordan claims that the number of conjugacy classes of transitive maximal soluble groups of degree $2^6 3^2 p$ is 8306).Again' the error in the first table is Likely to errors in this on too.Jordan (1872)caunts all the primitive groups of degrees **4** to **17**.His count Matches that of sims except that Jordan has one les for degrees **9**' **12** and **15**' eight less for degree **16** and two less For degree **17**.These errors are Pointed out by Miller(1894'4b'1895c'1897a'1897b and 1900a).Jordan(1874)states that every transitive group of degree **19** is either alternating' symetric or affine.This agrees With sim's list.Veronese(1885)determines all groups of degree up to **6**.

Miller(1983)cites several errors for degree **6**.Ask with(1890) and Cayley(1891) determine the groups of degree **6** and get the same answers.Their count of the intransitive ones is correct but undercount the transitive groups by three' as cole(1893a) points out' as kirkman(1862–3) had already (success fully' according to Miller)determined the transitinves ' degree **6** was then complete.Askwith(1890a) and Cayley(1891) also arrive at the same determination of the groups of degree **7**.They undercount both the transitive and intransitives by one' as cole(1893a) Points out.With kirkman's transitives and col's additional intransitive' Degree **7** was then complete .

2. Periliminares:

The first we determine the **JS**-imprimitives of $GL(4, p^k)$.

Recall that the number **JS**-maximals of $GL(2, p^k)$ as follows.

$$M_1(2,p^k) := GL(1,p^k) \;\; wr \;\; S_2, p^k \neq 2,$$
$$M_2(2,p^k) := C_{p^{2k}-1} \succ C_2,$$
$$M_3(2,p^k) := (C_{p^k-1} \mathrm{Y}\, Q_8) \;\; N\bar{O}(2,2), \qquad p^k \equiv 3(\mathrm{mod}4),$$
$$M_4(2,p^k) := (C_{p^k-1} \mathrm{Y}\, Q_8) \;\; N \;\; Sp(2,2), \quad p^k \equiv 1(\mathrm{mod}4).$$

Therefore the **Js**-imprimitives of $GL(4,p^k)$ are listed below.

$M_1(4,p^k) := GL(1,p^k) \;\; wr \;\; S_4, \quad p^k \neq 2,$

$M_2(4,p^k) := M_2(2,p^k) \;\; wr \;\; S_2,$

$M_3(4,p^k) := M_3(2,p^k) \;\; wr \;\; S_2, \quad p^k \equiv 3(\mathrm{mod}4),$

$M_4(4,p^k) := M_4(2,p^k) \;\; wr \;\; S_2, \quad p^k \equiv 1(\mathrm{mod}4).$

Now we listed the **JS**-primitives of $GL(4,p^k)$.

6.1-The JS-primitives of $GL(4,p^k)$.

There is one **JS**-primitive of $GL(4,p^k)$ whose unique maximal abelian normal subgroup has order $p^{4k}-1$, namely.

$$M_5(4,p^k) := C_{p^{4K}-1} \succ C_4,$$

the normaliser of a singer cycle.

There is one **JS**-primitive of $GL(4,p^k)$ whose unique maximal abelian normal subgroup has order $p^{2k}-1$, namely

$$M_6(4,p^k) := M_4(2,p^{2k}) \succ C_2, p \neq 2.$$

We think the filed of p^{2k} elements here as the filed of **2** by **2** matrices that is the linear span of our fixed singer cycle of $GL(2,p^k)$.

Now we come to the **JS**-primitives of $GL(4, p^k)$ whose unique maximal abelian normal subgroup is the scalar group.Let M be such a group. There are three cases examine.

6.1.1-Cas1:

Assume that $p^k \equiv 3(\mathrm{mod}4)$ and that the Fitting subgroup F of M is $C_{p^k-1} \mathrm{Y}\, D_8 \mathrm{Y}\, D_8.$ Then $\frac{M}{F}$ is isomorphic to a completely reducible maximal soluble subgroup of $O^+(4,2)$ that dosenot fix any non-zero isotropic subspace of the natural module for $O^+(4,2)$. However, $O^+(4,2)$ is isomorphic to $S_3 \; wr \; S_2$, so inparticular it is soluble. Therefore $\frac{M}{F} = O^+(4,2)$. So in this case we find exactly one **JS**-primitive, $M_7(4,p^k) := (C_{p^k-1} \mathrm{Y}\, D_8 \mathrm{Y}\, D_8) \;\; \mathrm{N}\, O^+(4,2).$

6.1.2-Case2:

Assume that $p^k \equiv 3 (\mathrm{mod}\, 4)$ and that the Fitting subgroup F of M is $C_{p^k-1} \mathrm{Y} D_8 \mathrm{Y} Q_8.$ Then $\frac{M}{F}$ is isomorphic to a completely reducible maximal soluble subgroup of $O^-(4,2)$ that dose not fix any non-zero isotropic subspace of the natural module for $O^-(4,2)$.

Threre is a unique $O^-(4,2)$-conjugacy class of such groups and that each group in this class is isomorphic to Hol (C_5).Therefore $\frac{M}{F}$ = $Hol\,(C_5)$.So in this case we find exactly on **JS**-primitive'

$M_8(4, p^k) := (C_{p^k-1} \mathrm{Y} D_8 \mathrm{Y} Q_8) \mathrm{N}\ Hol(C_5).$

6.1.3-Case3:

Assume that $p^k \equiv 1 (\mathrm{mod}\, 4).$ Therefore we can write the Fitting subgroups F of M as $C_{p^k-1} \mathrm{Y} D_8 \mathrm{Y} D_8.$

Then $\frac{M}{F}$ is isomorphic to a completely reducible maximal soluble subgroup of $Sp(4,2)$ that dosenot fix any non-zero isotropic subgroup of the natural module for $Sp(4,2)$.

Every such subgroup is $Sp(4,2)$-conjugacy to either $O^+(4,2)\ or\ Hol(C_5).$ So in this case we find exactly two **JS**-primitives

$$M_9(4, p^k) := (C_{p^k-1} \mathrm{Y} D_8 \mathrm{Y} D_8) \mathrm{N}\ O^+(4,2),$$

$$M_{10}(4, p^k) := (C_{p^k-1} \mathrm{Y} D_8 \mathrm{Y} D_8) \mathrm{N}\ Hol(C_5).$$

Although $M_9(4, p^k)$ and $M_{10}(4, p^k)$ admit the same description as $M_7(4, p^k)$ and $M_8(4, p^k)$,respectively' they arise in different way and so are given different numbers. For example the **JS**-maximals of $GL(4,3)$ are $M_1, M_2, M_3, M_5, M_6\ and\ M_8.$ Of these $M_3, M_5, M_7\ \ and\ M_8$ are maximal soluble' while $M_1\ and\ M_6$ are conjugate to subgroups of $M_7,\ \ and\ M_2$ is a conjugate to a subgroup of $M_3.$ The following table show the **JS**-maximals of $GL(4, p^k),$ for p^k =**1**' **…**' **30**.

Table 1. The JS-maximals of $GL(4,p^k)$, for $p^k = 1,...30$.

$GL(4,1)$	M_1	M_2		M_4	M_5	M_6			M_9	M_{10}
$GL(4,2)$		M_2			M_5					
$GL(4,3)$	M_1	M_2	M_3		M_5	M_6	M_7	M_8		
$GL(4,4)$	M_1	M_2			M_5					
$GL(4,5)$	M_1	M_2		M_4	M_5	M_6			M_9	M_{10}
$GL(4,6)$	M_1	M_2			M_5					
$GL(4,7)$	M_1	M_2	M_3		M_5	M_6	M_7	M_8		
$GL(4,8)$	M_1	M_2			M_5					
$GL(4,9)$	M_1	M_2		M_4	M_5	M_6			M_9	M_{10}
$GL(4,10)$	M_1	M_2			M_5					
$GL(4,11)$	M_1	M_2	M_3		M_5	M_6	M_7	M_8		
$GL(4,12)$	M_1	M_2			M_5					
$GL(4,13)$	M_1	M_2		M_4	M_5	M_6			M_9	M_{10}
$GL(4,14)$	M_1	M_2			M_5					
$GL(4,15)$	M_1	M_2	M_3		M_5	M_6	M_7	M_8		
$GL(4,16)$	M_1	M_2			M_5					
$GL(4,17)$	M_1	M_2		M_4	M_5	M_6			M_9	M_{10}
$GL(4,18)$	M_1	M_2			M_5					
$GL(4,19)$	M_1	M_2	M_3		M_5	M_6		M_8		
$GL(4,20)$	M_1	M_2			M_5		M_7			
$GL(4,21)$	M_1	M_2		M_4	M_5	M_6			M_9	M_{10}
$GL(4,22)$	M_1	M_2			M_5					
$GL(4,23)$	M_1	M_2	M_3		M_5	M_6	M_7	M_8		
$GL(4,24)$	M_1	M_2			M_5					
$GL(4,25)$	M_1	M_2		M_4	M_5	M_6			M_9	M_{10}
$GL(4,26)$	M_1	M_2			M_5					
$GL(4,27)$	M_1	M_2	M_3		M_5	M_6	M_7	M_8		
$GL(4,28)$	M_1	M_2			M_5					
$GL(4,29)$	M_1	M_2		M_4	M_5	M_6			M_9	M_{10}
$GL(4,30)$	M_1	M_2			M_5					

Now we determind the primitive and imprimitive soluble subgroups of $GL(4,2)$.

6.2-The imprimitive soluble subgroups of $GL(4,2)$.

By above table there is one **JS**-imprimitives of $GL(4,2)$, namely $M_2 := GL(2,2) \ wr \ S_2$ or $M_2 := M_2(2,2) \ wr \ S_2$.

A polycyclic presentation for M_2 is:

$$\{a,b,c,d,e \mid a^2 = 1,$$
$$b^a = d, b^2 = 1,$$
$$c^a = e, c^b = c^2, c^3 = 1,$$
$$d^a = b, d^b = d, d^c = d, d^2 = 1,$$
$$e^a = c, e^b = e, e^c = e, e^d = e^2, e^3 = 1\}.$$

6.2.1-Theorem:

A complete and irredundant list of $GL(4,2)$-jugacy class representatives of the irreducible subgroups of M_2 is: $<a,b,c,d,e>, <ab,c,e>, <a,bd,c,e>, <a,c,e>$.

proof:

See short (1992 ' 7.2 ' p.85).

By use from above table ' obviously that the **JS**-imprimitives of $GL(4,3)$ are $M_1(4,3), M_2(4,3)$ and $M_3(4,3)$. There are exactly **18** $GL(4,3)$-cojugacy classes of irreducible soluble subgroups whose gurdion is M_1.**33** $GL(4,3)$-conjugacy classes of irreducible soluble subgroups whose guardian is M_2 and **14** $GL(4,3)$-conjugacy classes of irreducible soluble subgroups whose gardian is M_3.Therefore there are 65 $GL(4,3)$-conjagacy classes of imprimitives soluble subgroups. We have picked exactly one representative from each of these classes. The following table details how many groups of each order there are in this set of representatives.(see short(1992 ' 7.3 ; p.86-93)).

Table 1. The imprimitive soluble subgroups of $GL(4,3)$.

Order	16	32	48	64	96	128	192	256	384	512	768	1152	2304	4608
Number	5	12	2	12	5	10	4	6	3	1	1	1	2	1

The **JS**-primitives of $GL(4,p^k)$, for $p^k = 3$ are $M_5, M_6, M_7 \ and \ M_8$ and for $p^k = 2$ is M_5.

Now we determine the primitive subgroups of $M_5, M_6, M_7 \ and \ M_8$.

6.4-The primitive subgroups of $M_5(4,3)$.

Recall that $M_5(4,p^k) = C_{p^{4k}-1} \succ C_4$, the normaliser of a singer cycle. A polycyclic presentation for M_5 is :

$$< a,b \mid a^4 = 1 \quad ,$$
$$b^a = b^{p^k}, b^{p^{4k}-1} = 1 > .$$

6.4.1-Theorem:

A complete and irredundant list of $GL(4,2)$-conjugacy class representatives of the primitive subgroups of $M_5(4,2)$ is:

$$<a,b>,<a^2,b>,<a,b^3>,<a^2,b^3>,<b>,<b^3>.$$

proof:

See short (1997, 8.5, pp.95-100).

Thus the six groups listed in the theorem are pairwise non-isomorphic, and so no two of them can be conjugate in $GL(4,2)$.

Therefore there are **10** $GL(4,3)$-conjugacy classes of irreducible soluble subgroups:**4** consist of imprimitive groups and **6** of primitives ones.

6.4.2-Theorem(short, 1992, 8.5, pp.100):

There are exactly **21** $GL(4,3)$-conjugacy classes of primitive soluble subgroups that have $M_5(4,3)$ as their guardian.

Thus by use above theorem we have obtained a complete and irredundant set of representatives of these classes.

6.5-A generating set for $M_6(4,p^k)$.

Recall that $M_6(4,p^k)=M_4(2,p^{2k})\succ C_2$. We already saw a polycyclic presentation for $M_4(2,p^{2k})$ in chapter **2**.From the proof of theorem **2.17**, we see that a extra element of order **2** needed to generate $M_6(4,p^k)$ can be chosen so that it acts

P-th poweringly on a generator of the center of $M_4(2,p^{2k})$, and trivially on each of the other members of our canonical generating set for $M_4(2,p^{2k})$. Therefore we have.

6.5.1-Proposition: $M_6(4,p^k)=M_2(2,p^k)\otimes M_i(2,p^k)$, where i is **3** or **4** according as p^k is congruent to **3** or **1** modulo **4**, respectively.

Proof:

Set $G:=M_2(2,p^k)\otimes M_i(2,p^k)$. Then $|G|=48(p^{2k}-1)$. Let A be our fixd singer cycle of $GL(2,p^k)$. It is not difficult to see that $A\otimes I_2$ is the unique maximal abelian normal subgroup of both G and $M_6(4,p^k)$. Therefore $G\le N_{GL(4,p^k)}(A)=M_6(4,p^k)$ (this last equality because $Sp(2,2)$ is soluble).Since $|G|=|M_6|$, the result follows.

It is now easly to write down a polycyclic presentation for $M_6(4,p^k)$.

6.6-The generating set for $M_7(4,p^k)$.

Recall that $M_7=(C_{p^k-1}\,\mathrm{Y}\,D_8\,\mathrm{Y}\,D_8)\ \mathrm{N}\ O^+(4,2)$. Let F be the Fitting subgroup of M_7. We could find a generating set by the methods given in chapter **2**. The following structure theorem yields a much simpler way.

6.6.1-Theorm:

M_7 is conjugate to $(M_3(2,p^k)\otimes M_3(2,p^k))\succ S_2$, where the non-trivial element of the S_2 is the permutation matrix that interchanges the tensor factors.

Proof:

Let G be the group $(M_3(2,p^k)\otimes M_3(2,p^k))\succ S_2,$ as defined in the statement of the theorem. Them $Fit(G)$ is the tensor square of $Fit(M_3(2,p^k)),$ and so $Fit(G)\cong C_{p^k-1}\curlyvee Q_8 \curlyvee Q_8.$ Since Q_8 is absolutely irreducible in $GL(2,p^k),$ it follows from the outer Tensor product Theorem that $Fit(G)$ is irreducible.Then by Theorem **4.3** $Fit(G)$ is conjugat to F .

So G is conjugate to a subgroup of the normaliser in $GL(4,p^k)$ of $F,$ namely $M_7.$ Since $|G|=|M_7|,$ the result follows.

We now redefine $M_7(4,p^k)$ to be $(M_3(2,p^k)\otimes M_3(2,p^k))\succ S_2.$

It is then easly to write down a polycyclic presentation for $M_7.$

Note that

$$M_7 \cong \begin{cases} C_{p^k-1}\curlyvee\ (GL(2,3)\ cr\ S_2),\ if\ \ p^k\equiv 3(\mathrm{mod}8), \\ \\ C_{p^k-1}\curlyvee\ (BO\ cr\ S_2),\qquad if\ p^k\equiv 7(\mathrm{mod}8), \end{cases}$$

where by $G\ cr\ T$ we mean the crown product of the group G and the transitive permutation groups **T**.In our case the crown product can be defined (abstractly) as the quotient of the wreath prooduct by the unigue diagonal central subgroup of the base group.

Obviously it is important to find the primitive subgroups of $GL(2,3)\ cr\ S_2$ and $BO\ cr\ S_2.$ Let N be either of these groups' let a be the centre of N (the scalar subgroup of order2) and let E be the Fitting subgroups of N . Then $\frac{E}{A}$ is symplectic space on which $\frac{N}{E}$ acts in the natural way as $O^+(4,2).$ The following theorem identifies those primitive subgroups of N which contain E .

6.6.2-Theorem:

Let G be a subgroup of N that contains E . Then G is imprimitive if and only if $\frac{G}{E}$ normalises amaximal isotropic subspace of $\frac{E}{A}$.

Proof: see short (1992' 8.3' pp.95-97)

6.7-The generating set for $M_8(4,p^k).$

Let F be the field of p^k elements' when $p^k\equiv 3(\mathrm{mod}4).$ we construct a generating set for M_8 by the following methods.

Let z be a generator for the scalar group' and define $u_1,v_1,u_2\ and\ v_2$ by

$$u_1:=I_2\otimes\begin{bmatrix}0&1\\1&0\end{bmatrix},u_2:=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\otimes I_2,v_1:=I_2\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix},v_2=\begin{bmatrix}\alpha&\beta\\\beta&-\alpha\end{bmatrix}\otimes I_2,$$

Where' α and β belong to the prime subfield of F and $\alpha^2+\beta^2=-1.$

Then $\{u_1, v_1, u_2, v_2, z\}$ generates the Fitting subgroup F of M_8. To extend this set to a generating set for M_8 ' we first require a generating set for $Hol(C_5)$. Actully' it is more convenient to choose a generating set for $O^-(4,2)$. we choose this set to consist of the three matrices $f\rho$, $g\rho$ and $h\rho$ defined below

$$f\rho := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g\rho := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$h\rho := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

setting $b\rho := h\rho f\rho$ and $a\rho := (g\rho f\rho)^2 b\rho g\rho$ ' we find that $a\rho$ has order 4, $b\rho$ has order 5' and $(b\rho)^{a\rho} = (b\rho)^2$, so that $< a\rho, b\rho > \cong Hol(C_5)$. There are tow

resons we work with $f\rho, g\rho$ *and* $h\rho$. One resons is that each is an involution; this means that the solutions to the liner equations we must solve are less complicated .The other reson is that each of these matrices has mainly zeros in its last colum; experience shows that calculations not involving v_2 are much simpler.It is also useful that the first three rows of $g\rho$ and $h\rho$ are the same . In this case' there exists a matrix f of $GL(4, F)$ such that

$u_1^f = \lambda_1 u_1$,
$v_1^f = \mu_1 u_1 v_1 v_2$,
$u_2^f = \lambda_2 u_1 u_2$,
$v_2^f = \mu_2 v_2$.

Where the λ_i .And μ_j are scalars.Setting $\lambda_1 = \lambda_2 = \mu_1$ *=1 and* μ_2 *=-1* .
We find that one such matrix is

$$f := 2^{-1} \begin{bmatrix} \beta & 1-\alpha & -\beta & 1+\alpha \\ -1-\alpha & -\beta & -1+\alpha & \beta \\ -\beta & 1+\alpha & \beta & 1-\alpha \\ -1+\alpha & \beta & -1-\alpha & -\beta \end{bmatrix}.$$

Then f has determinat **1** ’ and its square is $-I_4$.
Let δ be an element of F such that

$$\delta^2 = \begin{cases} -2 & if \;\; p^k \equiv 3(\mathrm{mod}\, 8) \quad , \\ 2 & if \;\; p^k \equiv 7(\mathrm{mod}\, 8) \quad ; \end{cases}$$

Then’ there exists a matrix g of $GL(4,F)$ such that

$u_1^g = \lambda_1 v_1,$

$v_1^g = \mu_1 u_1,$

$u_2^g = \lambda_2 u_2,$

$v_2^g = \mu_2 v_2.$

Where the λ_i and μ_j are scalars.Seting $\lambda_1 = \mu_1 = -1$ and $\lambda_2 = \mu_2 = 1,$
we find that one such matrix is

$$g := \delta^{-1} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}.$$

Then g has determinat **1**’ and its square is $-I_4 \;\; or \;\; I_4$’ acording as p^k is congruent to **3** or **7** modulo **8**.Then ’ there exists a matrix h of $GL(4,F)$ such that

$$u_1^h = \lambda_1 v_1 \quad ,$$
$$v_1^h = \mu_1 u_1 \quad ,$$
$$u_2^h = \lambda_2 u_2 \quad ,$$
$$v_2^h = \mu_2 u_2 v_2 .$$

Where the λ_i and μ_j are scalars.Setting $\lambda_2 = -1 \;\; and \;\; \lambda_1 = \mu_1 = \mu_2 = 1,$
we find that one such matrix is

$$h := 2^{-1} \begin{bmatrix} \alpha-\beta & \alpha+\beta & \alpha-\beta & \alpha+\beta \\ \alpha+\beta & -\alpha+\beta & \alpha+\beta & -\alpha+\beta \\ \alpha-\beta & \alpha+\beta & -\alpha+\beta & -\alpha-\beta \\ \alpha+\beta & -\alpha+\beta & -\alpha-\beta & \alpha-\beta \end{bmatrix}.$$

Then h has determinat **1**’ and its square is $-I_4$.
Now set $b := hf$ ’ Then

$$b = 2^{-1}\begin{bmatrix} -\alpha-\beta & \alpha-\beta & -\alpha-\beta & \alpha-\beta \\ \alpha-\beta & \alpha+\beta & \alpha-\beta & \alpha+\beta \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

Also, b has determinant 1 and order 5.The action of b on F is given by the matrix

$$b\rho = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Now set $a := (gf)^2 bg$.Then

$$a = \delta^{-3}\begin{bmatrix} 1+\alpha-\beta & 1+\alpha+\beta & -1-\alpha-\beta & 1+\alpha-\beta \\ -1+\alpha+\beta & 1-\alpha+\beta & -1+\alpha-\beta & -1+\alpha+\beta \\ -1-\alpha+\beta & -1-\alpha-\beta & -1-\alpha-\beta & 1+\alpha-\beta \\ 1-\alpha-\beta & -1+\alpha-\beta & -1+\alpha-\beta & -1+\alpha+\beta \end{bmatrix}.$$

Also, a has determinate **1** and its fourth power is -I_4.Furthermore, $b^a = b^2$. The action of a on F is given by the matrix

$$a\rho = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Finally we have that $M_8 = < a, b, u_1, v_1, u_2, v_2, z >$.

Set $N := < a, b, u_1, v_1, u_2, v_2 >$.It is clear that $N = M_8 \bigcap SL(4, F)$ and that $M_8 = N \;\; Y < z >$.The given element, whitout z, from a polycyclic generating sequence for N, and the corresponding relations are:

$$a^4 = -I_4 \quad ,$$

$$b^a = b^2 \quad , \quad b^5 = I_4 \quad ,$$

$$u_1^a = -v_1 \quad , \quad u_1^b = u_1 v_1 v_2 \quad , u_1^2 = I \quad ,$$

$$v_1^a = u_1 v_1 u_2 \; , \quad v_1^b = u_1 \quad , v_1^{u_1} = -v_1 \quad , v_1^2 = I_4 \quad ,$$

$$u_2^a = -v_1 v_2 \quad , \quad u_2^b = -u_1 u_2 \quad , u_2^{u_1} = u_2 \quad , u_2^{v_1} = u_2 \quad , u_2^2 = -I_4 \quad ,$$

$$v_2^a = v_1 u_2 v_2 \; , \quad v_2^b = -u_1 u_2 v_2 \quad , v_2^{u_1} = v_2 \quad , v_2^{v_1} = v_2 \quad , v_2^{u_2} = -v_2 \quad , v_2^2 = -I_4 .$$

Note that each of these relations is independent of the value of p^k modulo **8**.
It can be checked via **CAYLEY** that N dosenot split over $N \cap F$.

6.8-Some primitive subgroups of $M_6(4, p^k)$

Recall that $M_6(4, p^k) = M_2(2, p^k) \otimes M_i(2, p^k),$ where **i** is **3** or **4**' acording as p^k is congruent to **3** or **1** modulo **4**' respectively' Also recall that

$$M_i(4, p^k) \cong C_{p^k-1} \mathrm{Y} \begin{cases} BO & if \quad p^k \equiv \pm 1 (\mathrm{mod} 8) \ , \\ GL(2,3) & if \quad p^k \equiv 3 (\mathrm{mod} 8) \ , \\ NS & if \quad p^k \equiv 5 (\mathrm{mod} 8) \ . \end{cases}$$

Therefore' since the scalar group is the subgroup that is amalgamated in the tensor product' we may write

$$M_6(4, p^k) \cong (C_{p^k-1} \succ C_2) \mathrm{Y} \begin{cases} BO & if \quad p^k \equiv \pm 1 (\mathrm{mod} 8), \\ GL(2,3) & if \quad p^k \equiv 3 (\mathrm{mod} 8), \\ NS & if \quad p^k \equiv 5 (\mathrm{mod} 8). \end{cases}$$

Then subgroup amalgamated in this central product of order **2**' except when $p^k \equiv 5 (\mathrm{mod} 8),$ in which case it has order **4**. In this central decomposition of $M_6,$ denote the first central factor by X and the second by Y .
In this section we determine some primitive subgroups of $M_6(4, p^k)$ by using our knowledge of X, Y and of the subgroups of central products.

6.8.1-Proposition:

Every primitive subgroup of $M_6(4, p^k)$ contains the scalar matrix $-I_4$.

Proof:

The proof when $p^k \equiv \pm 1 (\mathrm{mod} 8)$ is sufficient to indicate the method. Let G be a subgroup of $M_6(4, p^k)$ that dosenot contain $-I_4$.
By examining the Burnside inclusion diagram of BO' we see that $G \cap Y$ must be of order **1** or **3**.Therefore $G \leq X \otimes C_6$. Observe that $X \otimes C_6 \cong X \times C_3$ .If $p = 3$' then this group is reducible' and if $p \neq 3,$ then it is not primitive' because it has a
non–cyclic abelian normal **3**-subgroup.
We now specialise to the case $p^k \equiv 3 (\mathrm{mod} 8)$.Then $Y \cong GL(2,3).$
Let G be a primitive subgroup of $M_6(4, p^k)$ .Then $G \geq X \cap Y,$ and so by theorem **3.1.5**' we can parametrise G by the triple $(X_1/X_0, Y_1/Y_0, \theta),$

Where $X_1 = X \cap GY,\ X_0 = X \cap G, Y_1 = XG \cap Y,\ \ Y_0 = G \cap Y$ and θ is an isomorphism from X_1/X_0 to Y_1/Y_0 .Since G is primitive, it is clear(from elementary propertise of the tensor product)that both X_1 and Y_1 must be primitive.The primitive subgroups of X were discussed in section **2**, and the primitive subgroups of Y were discussed in section **3**.

6.8.2-Lemma:

The only possibilities for the pair (Y_1, Y_0) are $(GL(2,3),GL(2,3)),(GL(2,3),SL(2,3)),(GL(2,3),Q_8),(SL(2,3),SL(2,3))\ \ and\ \ (SL(2,3),Q_8)$.

Proof: By use from the burnside inclusion diagram of $GL(2,3)$ yeilds that, if Y_1 did not contain $SL(2,3)$, Then G would be a subgroup of $X \otimes SD_{16}\ \ or\ \ X \otimes D_{12}$.

The first of these obviously has a non-cyclic abelian normal 2-subgroup and so cannot be primitive.If $p = 3$, then $X \otimes D_{12}$ is reducible, and if $p \neq 3$, then $3 1 (p^{2k} - 1)$ and so $X \otimes D_{12}$ can not be primitive, because it has a non-cyclic abelian normal 3-subgroup.Hence $Y_1 \geq SL(2,3)$. Since X is metacyclic, it follows that X_1/X_0 is metacyclic.Therefor Y_1/Y_0 must be metacyclic too.

6.8.3-Teorem:

Let G be a subgroup of M_6 such that $G \cap Y \geq SL(2,3)$. If B is an abelain normal subgroup of G, then $B \leq X_0$.

Proof:

Without loss of generality we can assume that $B \geq < -I_4 >$.

Clearly $B \cap Y$ is an abelian normal subgroup of $G \cap Y$, which is $SL(2,3)\ \ or\ \ GL(2,3)$.

Therefore $B \cap Y = < -I_4 >$. Also $G \cap Y$ must normalise $XB \cap Y$. The only subgroups of $GL(2,3)$ that are normalised by $SL(2,3)$ are the normal subgroups of $GL(2,3)$ .Since B is abeline, we conclude that $XB \cap Y$ is $< -I_4 >$ or Q_8.

Since B is normal in G, we have from by Theorem **3.1.5** that $(X \cap BY)/(X \cap B) \cong_{\Omega} (XB \cap Y)/(B \cap Y)$, Where Ω is the set of outomorphism of $X \otimes Y$
induced by the elements of G acting by conjugation.

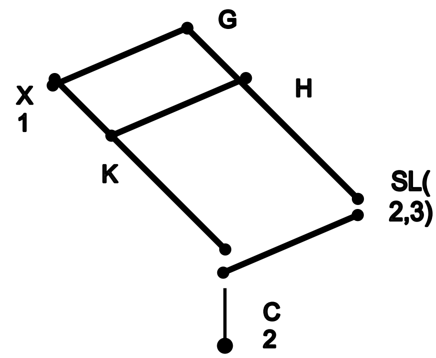
Since the **3**-elements of $SL(2,3)$ act trivially on X but non-trivially on $Q_8 / < -I_4 >$, we conclude that B cannot be a diagonal subgroup of $X \otimes Y$,
and thus $B \leq X$.

6.8.4-Theorem:

If X_1 is primitive subgroup of X, then $X_1 \otimes SL(2,3)$ is a primitive subgroup of M_6.

Proof: See short(1992, theorem 8.6.4, pp.102-103)

Let $G := X_1 \otimes SL(2,3) \quad and \quad H = K \otimes SL(2,3),$ Where $< -I_2 > \leq K \leq X_1$. Then by proof of above theorem, we have the Figure following



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6.8.5-Corollary 1:

If X_1 is a primitive subgroup of X, then $X_1 \otimes GL(2,3)$ is a primitive subgroup of M_6.

6.8.6-Theorem K:

If X_1 is a primitive subgroup of X, then the diagonal between $X_0 \otimes SL(2,3) \quad and \quad X_1 \otimes GL(2,3)$ is primitive.

Proof:

See short(1992, theorem 8.6.5, pp.103-104)

Now we specialise to the case $p^k = 3.$ We have

$$M_6(4,3) = M_2(2,3) \otimes M_3(2,3) = SD_{16} \otimes GL(2,3).$$

The primitive subgroups of SD_{16} are SD_{16}, Q_8 and C_8. When $X_1 = X_0$, we get six group, all of which are primitive by theorem K and it corollary when X_1 / X_0 has

Order **2**, we get five groups, all of which are primitive by theorem K.

ISOTEST shows that among these eleven groups there are ten distinct isomorphism types of sylow 2-subgroups .The two groups whose sylow **2**-subgroups are isomorphic have derived groups of deffirent orders.Therefore these eleven groups are pairwise non-isomorphic.None of these groups is isomorphic to a subgroup of $M_5(4,3)$ because that group is metacyclic.So there are exactly $11 \;\; GL(4,3)$

-conjugacy classes of primitive soluble subgroups whose gurdian is M_6.

6.9-The primitive subgroups of $M_7(4, p^k)$ ***when*** $p^k \equiv 3 (\bmod 8).$

we assume that $p^k \equiv 3 (\bmod 8)$ and F is the field of $p^k -$ elements.Also, We denote $GL(4, F)$ by L .We established in section **3** that

$$M_7(4, p^k) = C_{(p^k-1)/2} \times (GL(2,3) \;\; cr \;\; S_2).$$

Denote by N ***the second direct factor in the decomposition.We must find the primitive subgroups of*** N ***.For this work, we have:***

6.9.1-Proposition 1:

Let E be a finite field, and let A and G be subgroups of $GL(n,F)$ With A abelian G primitive soluble.If G normalises A, then G is contained in a primitive maximal soluble subgroup of $GL(n,E)$ whose unique maximal abelian normal subgroup contains A.

Proof:

Denote $GL(n,E)$ by L.Since A is an abelian normal subgroup of the primitive Soluble group GA, it follows that A is homogeneous and cyclic.Consequently, $C_L(A)=GL(\frac{n}{m},K),$ for some divisor m of n, and where K is a field extension of E of degree m.Then $B:=C_L(C_L(A))$ is isomorphic to the multiplicative group of K, and cotains A.Since G normalises A, it follows that G normalises B.Let M be a maximal soluble subgroups of $N_L(B)$ that contains G.Then by Theorem **2.5** follows that M is also a maximal soluble subgroup of L.

6.9.2-Main Theorem:

If G is a primitive subgroup of N that is not conjugate to a subgroup of $M_5(4,F)$ or $M_6(4,F)$, then G contains $Fit(N)$.

Proof:

(L.G.kovacs)Let G be a primitive subgroup of N that is not conjugate to a subgroup of $M_5\ \ or\ \ M_6$.The **2**.Subgroups of L are not Primitive, and thus G contains a non-trivial 3-element.The sylow **3**-subgroups of N are elementary abelian of order **9**.Since they are not cyclic, it follows that $O_3(G)$ is of order 1 or 3.Suppose the latter were true.Since $O_3(G)$ would not be scalar, it would follow from the proposition 1 that G is conjugate to a subgroup of a **JS**-maximal whose uniqe maximal abelian normal subgroup is not scalar.The only such groups are $M_5\ \ and\ \ M_6$, so we have reached a contradiction.Therefore $O_3(G)=1$.We have shown that $Fit(G)$ is a **2**-group.Since $O^+(4,2)$ (that is $N/Fit(N)$) has no non-trivial normal **2**-subgroups, it follows that $Fit(G)=G\bigcap Fit(N).$

By corollary(iF G is a primitive subgroup of $GL(n,F)$, and N is a nilpotent normal subgroup of G, then every abelian characteristic subgroup of N is cyclic).every abelian chareristic subgroup of $Fit(G)$ is cyclic.If a subgroup of $D_8 \Upsilon D_8$ has this property, Then it is isomorphic to one of $1,C_2,C_4,D_8,Q_8,C_4\Upsilon D_8$ and $D_8\Upsilon D_8$.Since G is not a **2**-group and $Fit(G)$ is a **2**-group, it follows that **Out**($Fit(G)$) is not a **2**-group.Therefore $Fit(G)$ is isomorphic to $Q_8,C_4\Upsilon D_8\ \ or\ \ D_8\Upsilon D_8$.Suppose the second were the case.Then $Z(G)$ would be cyclic of order **4**, yet not scalar.Then by proposition **1** G would be canjugate to a subgroup of $M_5\ \ or\ \ M_6$,a contradiction.Now suppose that $\mathrm{Fit(G)}\cong Q_8$. Then $G/Fit(G)$ is $C_3\ \ or\ \ S_3$ (because **Out** (Q_8) is isomorphic to S_3).

There are only two subgroups of $\mathrm{Fit(N)}$ that are isomorphic to Q_8, and so G normalises both of them.Therefore G is a subgroup of the base group $GL(2,3)\ \Upsilon\ GL(2,3)$ of N.Denote these central factors by X and Y.(Refer to Figure for the Burnside inclusion diagram of $GL(2,3)$).With out loss of generality we can assume that

$Fit(G) = Fit(Y)$.Since $G/Fit(G)$ has order dividing 6' therefore $X \cap GY$ has order dividing **12**.(By Theorem **3.1.5**).If this group were not $<-I_4>$, then G would normalise and abelian subgroup of X that was not scalar-this would imply that G were conjugate to a subgroup of $M_5\ or\ M_6$, a contradiction.On the other hand' if $X \cap GY$ were equal to $<-I_4>$, then G would be a subgroup of Y and so reducible' another contradiction.Hence $Fit(G) \cong Q_8$. This completes the proof.

It remains only to consider the subgroups of N that contain $Fit(N)$.By use from the subgroups of $O^+(4,2)$ we find that there are 13 conjugacy classes of primtive subgroups of N that contain $Fit(N)$. Five of this classes contain subgroups of M_5 or M_6 ; The other eight donot.Thus there are eight $GL(4,p^k)$-conjugacy classes of subgroups of N whose gurdian is M_7.

In particular there are eight $GL(4,3)$-conjugancy classes of primitive soluble subgroups whose guardian is M_7.

6.10-The primitive subgroups of $M_8(4,p^k)$.

Recall that of section **4**

$$M_8 = C_{(p^k-1)/2} \times ((D_8 Y Q_8)\quad N\quad Hol(C_5)).$$

Denote by N the second direct factor in this decomposition.We wish to find the primitive subgroups of N .The 2-subgroups of N are not primitive. Using the CAYLEY function lattice we find that there are exactly seven conjugacy classes of subgroups of N which are not 2-groups.

The sub-diagram generated by these seven classes in the Burnside inclusion diagram is shown in Figure **6.10.1**.

640

320

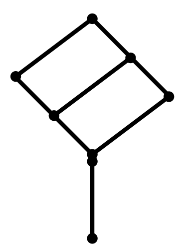
40

I_5^4

160

C_{10}

C_5



Figuare 6.10.1:

The Burnside inclusion diagram of the non **2**-subgroups of N .

Note that each subgroup of N of order **40** is metacyclic.If subgroup is primitive' then by theorem **5.3.1** (from' L.G.kovacs) it is conjugate to a subgroup of $M_5(4,p^k)$. Consequently' if G is a primitive subgroup of M_8 that is a subdirect product of group of scalars and a subgroup of N of order dividing **40**'

then the guardian of G is M_5.This leaves just three subgroups of N to consider.By use of the subgroups of $O^-(4,2)$,We known the cyclic subgroups of $O^-(4,2)$ of order **5** do not fix any non-zero isotropic subspace of the nutural module for $O^-(4,2)$.Therefore by theorem **2.35**' each of the three remaining groups is primitive.

In each of these three groups' we need to know the normal subgroups with cyclic quotients of odd order .The information in table **6.10.1** shows that there are very few such normal subgroups.

Table 6.10.1. The cyclic quotinets of some primitive subgroups of N .

G	G'	G/G'
$(D_8\ Y\ Q_8)\ N\ HOL(C_5)$	$(D_8\ Y\ Q_8)\ N\ C_5$	C_4
$(D_8\ Y\ Q_8)\ N\ D_{10}$	$(D_8\ Y\ Q_8)\ N\ C_5$	C_2
$(D_8\ Y\ Q_8)\ N\ C_5$	$D_8\ Y\ Q_8$	C_5

We could now write down the primitive subgroups of $M_8(4, p^k)$ that are not conjugate to subgroups of $M_5(4, p^k)$.It remains only to show that none of these groups is conjugate to a subgroup of $M_6(4, p^k)$ or $M_7(4, p^k)$.

This follow from the fact that neither $|M_6 : Fit(M_8)|\ \ nor\ \ |M_7 : Fit(M_7)|$ has 5 as a divisor.In particular’ there are exactly three $GL(4,3)$-conjugacy classes of primitive soluble subgroups whose guardian is M_8.

6.11-Result:

There are 108 $GL(4,3)$-conjugacy classes of irreducible soluble subgroup:65 consist of imprimitive groups and **43** of primitive ones.

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